# Two-dimensional incompressible magneto-hydrodynamic flow across an elliptical solenoid

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SUMMARY

An analysis has been made of the two-dimensional flow of an incompressible constant-conductivity fluid through an elliptically shaped solenoid containing a constant magnetic field directed normal to the flow plane. The effect of both Hall current and ion slip has been included in the generalized Ohm's law used for the fluid. The analysis is based on a perturbation procedure in two parameters, one being the magnetic Reynolds number $R_m$ and the other the ratio $S$ of magnetic force per unit area to dynamic pressure. Calculations have been carried to the first order in each parameter, and closed-form analytic expressions have been obtained for the force and moment on the solenoid, the current density, stream function, magnetic field and other pertinent physical quantities.

It was found that, to the zeroth order, there is a force but no moment on the solenoid. To the first order in $S$, where the flow field is modified but the magnetic field is not, there is a moment and a force, the latter being anti-parallel to the zeroth order force. To the first order in $R_m$, where the magnetic field is modified but the flow field is not, there is a moment but no force. Thus, to the first order the lift to drag ratio is the same as in the zeroth order. Graphs which illustrate some of the effects of angle of attack, fineness ratio of the ellipse, Hall current and ion slip, on the forces and moments are presented.

SYMBOLS

| | | | |
|---|---|---|---|
| $A_m$ | constant in Fourier expansion of $(\partial\psi_i/\partial\eta)_e$, see (4.7), | $C_e$ | contour of elliptical solenoid, |
| $a, b$ | semi-major and semi-minor axes of ellipse, | $c$ | value of $\xi$ on elliptical solenoid, |
| $\mathbf{B}$ | magnetic field vector, | $D$ | drag, |
| $B^0$ | uniform magnetic field inside ellipse (dimensional), | $d$ | see (3.2), |
| $C_m$ | constant in Fourier expansion of $(\partial\psi_i/\partial\eta)_e$, see (4.7), | $\mathbf{E}$ | electric field vector, |
| | | $E$ | complete elliptic integral of the second kind, |
| | | $e$ | electronic charge, |

$\mathbf{F}$ force,
$h$ fineness ratio $b/a$ of ellipse
$I$ discontinuous factor which is unity inside ellipse, zero outside ellipse,
$\mathbf{j}$ current density vector,
$K_t$ constant in stream function $\psi'$, see (4.28),
$K$, $K_\xi$, $K_c$, $K_s$ constants defined below (4.32 b) and (5.9),
$\mathbf{k}$ unit vector normal to flow plane, forming third vector of a right-handed system,
$k$ Boltzmann's constant,
$L$ lift,
$\mathscr{L}$ reference length,
$l$ coordinate along free-stream direction,
$\mathbf{M}$ moment about centre of ellipse,
$m$ particle mass,
$n$ coordinate perpendicular to free stream direction,
$N$ wake half-width, see (4.19),
$N_j$ number density of $j$ species,
$P$ perimeter of ellipse,
$p$ fluid pressure,
$\mathbf{q}$ velocity vector,
$q^0$ free-stream speed (dimensional),
$Q_{jk}$ elastic collision cross-section for collision between species $j$ and $k$,
$R_m$ magnetic Reynolds number $\mu\mathscr{L}\sigma q^0$,
$R$ moment arm,
$\mathbf{r}$ position vector from centre of ellipse,
$S$ dimensionless parameter $\sigma\mathscr{L}(B^0)^2/\rho q^0$,
$s$ arc length along a contour,
$T$ temperature,
$\mathbf{U}$ constant vector, defined below (5.9),
$\mathbf{V}^0$ vector, see (4.32),
$\bar{\mathbf{V}}^0$ constant vector, defined below (5.9),
$W_c$, $W_v$ constants in vorticity jump, see (4.20 c),
$x, y$ coordinate along and normal to the major axis of the ellipse,
$z$ complex position vector, $z = x + iy$,
$\alpha$ angle of attack, measured between major axes of ellipse and free stream, see figure 1,
$\alpha$, $\beta$, $\gamma$, $\delta$ constants in the solution of Laplace's equation (with various sub- and superscripts),
$\Delta_m$ see (4.7 a),
$\epsilon$ eccentricity of ellipse, $\sqrt{(1-h^2)}$,
$\epsilon_{jk}$ friction coefficient between species $j$ and $k$, see (A 3),
$\zeta$ $\xi + i\eta$,
$\eta$ angular elliptic coordinate,
$\theta$ polar angle in $(x,y)$-plane,
$\kappa$ Hall coefficient, see (A 2 b),
$\lambda$ ion slip coefficient, see (A 2 c),
$\mu$ permeability,
$\xi$ radial elliptic coordinate,
$\rho$ fluid density (dimensional),
$\sigma$ electrical conductivity of fluid (dimensional),
$\phi$ electrostatic potential, $E = -\nabla\phi$,
$\psi$ stream function, $\mathbf{q} = -\mathbf{k} \times \nabla\psi$,
$\mathbf{\Omega}$ vorticity vector, $\mathbf{\Omega} = \nabla \times \mathbf{q}$.

*Subscripts*

$A$ applied,
$e$ on the elliptical contour,
$E$ electrons,
$i$ inside ellipse,

*Subscripts—continued.*

| | | | |
|---|---|---|---|
| $I$ | ions, | $w$ | in wake, |
| $N$ | neutral atoms or molecules, | $x, y$ | components of vector in $x$-, $y$-directions, |
| $l, n$ | components of vector in $l$-, $n$-directions, | 1 | unit vector, |
| $o$ | outside ellipse, | $\nu$ | outer normal to contour. |

*Superscripts*

| | | | |
|---|---|---|---|
| 0 | zero order, | $a$ | first order in $R_m$, |
| $'$ | first order in $S$, | $*$ | dimensional quantity. |

## 1. Introduction

The possibility that magnetohydrodynamic forces may be used advantageously in high speed flight has recently been suggested (Kantrowitz 1953; Patrick 1956; Rosa 1956). At the gas temperatures encountered at speeds corresponding to a re-entering satellite, air becomes a reasonably good conductor of electricity (Lamb & Lin 1957). If this air is allowed to flow through a magnetic field, drag, lift, or control forces may be obtained. It can be shown, on the basis of very rough order-of-magnitude arguments, that it may be possible to obtain larger forces with less heat transfer by using magnetohydrodynamic forces than would be obtained by the conventional method of using gas pressure acting on solid surfaces.

In order to make a more precise evaluation of the possibilities of magnetohydrodynamics in flight, it is necessary to develop methods for treating magnetohydrodynamic flow problems. The work reported here is intended as a step in this direction. Its aim was to analyse a physically realistic magnetohydrodynamic flow problem which maintains some of the features of hypersonic flight. However, some of the simplifications that have been introduced to obtain analytical solutions are such that the direct application of the results to flight is not possible. In particular, we will assume incompressible flow and uniform electrical conductivity in the entire flow field. While this is a case which is physically realizable in either subsonic flows of hot gas or liquid metal flows, it is not a good approximation to hypersonic flight. Nevertheless, it is hoped that the understanding gained will lead towards the solution of realistic flight problems.

For magnetohydrodynamic calculations associated with astrophysical conditions or with the conditions obtained in highly pinched discharges, the magnetic Reynolds number (see (2.9b)) is usually assumed to be very large. This is frequently stated more crudely as the assumption of infinite conductivity. This is justified in astrophysics because of the large lengths involved and in the pinched discharges because of the high electrical conductivity which is achieved at very high temperatures ($\sim 10^6\,^\circ$K).

For large magnetic Reynolds numbers the flow and the magnetic field are very closely coupled and the field lines may be considered almost rigidly attached to the fluid particles. The value of the magnetic Reynolds number that can be expected in flight is, however, of the order of unity or less. The magnetic field is then only slightly distorted by the flow. The general analysis in this paper is oriented towards small magnetic Reynolds numbers and the final calculation is a perturbation calculation based, in part, on an expansion in powers of the magnetic Reynolds number.

In the presence of a magnetic field, the relation between the current and applied electric field—Ohm's law—in a gas is considerably more complex than it is in a solid or liquid conductor. At low densities and high magnetic field strengths the electrons perform many revolutions in their helical orbits in the magnetic field between collisions. Under these conditions the current may be reduced and is not parallel to the electric field, as seen in coordinates moving with the gas. At still lower densities, and in a partially ionized gas, an appreciable difference between the mean velocity of the ions and the gas velocity arises. This results in a reduction of the effective conductivity of the gas. Both of these effects will probably be important under the conditions of interest for flight applications. The generalized form of Ohm's law utilized in this paper is valid for general values of the parameters describing these two effects and is restricted only by the assumption that the degree of ionization is small.

A type of configuration which might be of interest in hypersonic flight is an extended magnetic field such as that produced by a large ring current surrounding a relatively small body. Such a configuration would produce a drag of the order of magnitude of the dynamic pressure acting on an area comparable with the area enclosed by the ring. The area of solid surface exposed to high heat transfer rates is, however, comparatively small. In such a case the flow interacts principally with the magnetic field, and the disturbance due to the presence of the body and the field coils may be neglected to a first approximation.

In the present paper this type of configuration is idealized by considering a two-dimensional flow perpendicular to the axis of an infinitely long solenoid (figure 1). The solenoid is of elliptical cross-section with its major axis at an angle $\alpha$ to the flow direction. It has a uniform magnetic field normal to the flow plane inside the ellipse, and no field outside. We assume that the solenoid itself is completely transparent to the flow; that is, the current-carrying wires are thin enough not to affect the flow. The only interaction with the fluid is that caused by the magnetic field inside the solenoid. To further simplify the problem, we take the fluid to be incompressible, with constant conductivity.

The method of solution of the problem is based on a perturbation in two parameters. One, as mentioned above, is the magnetic Reynolds number which describes the perturbation of the magnetic field about the undisturbed field of the solenoid. The other is a parameter which describes the perturbation of the flow about the uniform flow which would exist in the absence of a magnetic field. This second perturbation procedure is similar to thin-aerofoil theory, in that it is a small disturbance of a uniform flow. The magnetohydrodynamic case is, however, not restricted to slender cross-sections of the solenoid since for weak magnetic fields the effect on the flow field will be small regardless of the shape. In connection with the remark about aerofoil theory it might be noted that, from an aeronautical point of view, the solenoid behaves as a 'magnetic aerofoil', disturbing the flow and producing a force and moment on itself, as does a solid wing.

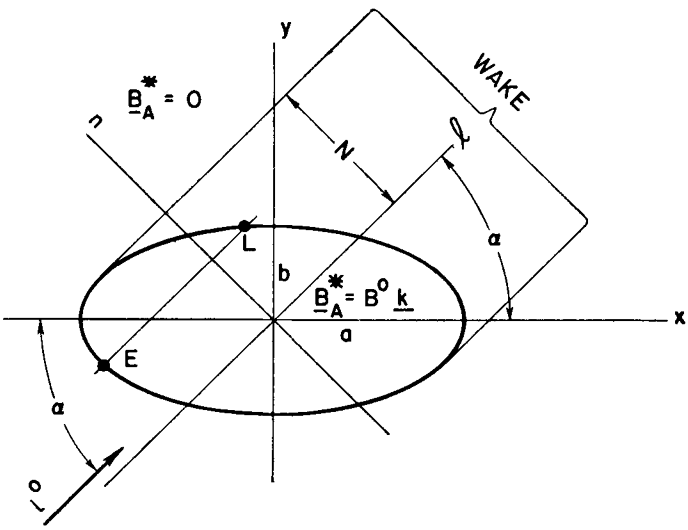


Figure 1. Notation and coordinate systems.

The first-order changes in both the flow and the magnetic field have been computed. The resulting forces and moments on the solenoid are also determined by considering the corresponding forces and moments due to the magnetic body forces on the fluid.

## 2. Basic equations

The basic equations for the steady motion (with velocity $\mathbf{q}^*$) of an incompressible fluid of density $\rho$ and constant conductivity $\sigma$ in the presence of a magnetic field $\mathbf{B}^*$ and electric field $\mathbf{E}^*$ are a combination of the Euler, continuity, and Maxwell equations. In rationalized mks. units they are

$$\rho(\mathbf{q}^* . \nabla)\mathbf{q}^* + \nabla^* p^* - \mathbf{j}^* \times \mathbf{B}^* = 0, \qquad \nabla^* . \mathbf{q}^* = 0, \qquad (2.1\,\text{a, b})$$

$$\nabla^* \times \mathbf{E}^* = 0, \qquad \nabla^* . \mathbf{B}^* = 0, \qquad \nabla^* \times \mathbf{B}^* = \mu \mathbf{j}^*, \qquad (2.2\,\text{a, b, c})$$

where $\mu$ is the magnetic permeability and $\mathbf{j}^*$ the current density. Equations (2.1 a, b) are the usual incompressible flow equations except for the presence of the term $\mathbf{j}^* \times \mathbf{B}^*$ which represents the body force exerted by the electromagnetic field on the fluid. (The remaining Maxwell equation which relates the divergence of the electric field to the net charge density may in all cases of magnetohydrodynamic interest be replaced by the quasi-neutrality condition which states that the electron and ion densities are equal at all points in the fluid*. This condition enters into the derivation of the generalized Ohm's law for the gas and is not required directly in the macroscopic equation.) For a partially ionized gas with a low degree of ionization, the current density is related to the electromagnetic and velocity fields by a generalized Ohm's law of the form (see Appendix A)

$$\mathbf{j}^* = \sigma[\mathbf{E}^* + \mathbf{q}^* \times \mathbf{B}^* - \kappa^* \mathbf{j}^* \times \mathbf{B}^* + \lambda^*(\mathbf{j}^* \times \mathbf{B}^*) \times \mathbf{B}^*]. \tag{2.3}$$

We will work with a dimensionless form of the basic equations obtained by introducing the following dimensionless variables:

$$\mathbf{q} = \frac{\mathbf{q}^*}{q^0}, \quad p = \frac{p^*}{\rho(q^0)^2}, \quad \mathbf{B} = \frac{\mathbf{B}^*}{B^0}, \quad \mathbf{E} = \frac{\mathbf{E}^*}{q^0 B^0}, \quad \mathbf{j} = \frac{\mathbf{j}^*}{\sigma q^0 B^0}. \quad (2.4\,\text{a, b, c, d, e})$$

Here $q^0$ and $B^0$ are the free stream velocity and the constant field inside the solenoid, respectively. The coordinates are also made dimensionless with a characteristic length $\mathscr{L}$. Then (2.1), (2.2) and (2.3) become

$$(\mathbf{q}\,.\,\nabla)\mathbf{q} \equiv \tfrac{1}{2}\nabla q^2 - \mathbf{q} \times \mathbf{\Omega} = -\nabla p + S\mathbf{j} \times \mathbf{B}, \quad \nabla\,.\,q = 0, \qquad (2.5\,\text{a, b})$$

$$\nabla \times \mathbf{E} = 0, \quad \nabla\,.\,\mathbf{B} = 0, \quad \nabla \times \mathbf{B} = R_m \mathbf{j}, \qquad (2.6\,\text{a, b, c})$$

$$\mathbf{j} = \mathbf{E} + \mathbf{q} \times \mathbf{B} - \kappa \mathbf{j} \times \mathbf{B} + \lambda(\mathbf{j} \times \mathbf{B}) \times \mathbf{B} \tag{2.7}$$

where $\mathbf{\Omega} \equiv \nabla \times \mathbf{q}$ and the dimensionless parameters $\kappa$ and $\lambda$ are defined by

$$\kappa = \kappa^* \sigma B^0, \qquad \lambda = \lambda^* \sigma (B^0)^2. \qquad (2.8\,\text{a, b})$$

The two dimensionless numbers $S$ and $R_m$ which appear are defined by

$$S \equiv \frac{\sigma \mathscr{L}(B^0)^2}{\rho q^0}, \qquad R_m \equiv \mu \sigma q^0 \mathscr{L}. \qquad (2.9\,\text{a, b})$$

$R_m$ is the magnetic Reynolds number and $S$ is a parameter which represents the ratio of electromagnetic body force per unit area to fluid dynamic pressure. Equation (2.6c) shows that the magnitude of $R_m$ determines the magnitude of the changes in the magnetic field due to the currents in the fluid.

The parameters that appear in the non-dimensional form of the equation depend, of course, on the method of non-dimensionalization. The non-dimensionalization of most of the parameters is fairly straightforward. There is, however, some freedom as far as the current is concerned. For small $R_m$ the non-dimensional current defined by (2.4e) will be of the order of unity. However, for large $R_m$ the flow and magnetic field adjust themselves so that the current is much less than $\sigma q^0 B^0$. In this case the magnetic field can change by its own order of magnitude in the characteristic length. The current should, therefore, be non-dimensionalized with respect

* For a discussion of this point see, for example, Cowling (1953).

to the magnetic field gradient $B^0/\mu\mathscr{L}$. If this is done the parameters entering into the non-dimensional equations are $R_m$ and $S/R_m$. It would seem, therefore, that the appropriate parameter for measuring the electromagnetic forces is $S/R_m$ for large $R_m$, and $S$ for small $R_m$. The parameter $S/R_m$ is frequently found in the literature because of the preponderance of interest in the large-$R_m$ case. In the present case of small $R_m$, we will use $S$.

Equation (2.7) can be solved for $\mathbf{j}$ by expanding the triple vector product, forming $\mathbf{j}\times\mathbf{B}$, and then substituting the resulting expression back into (2.7). This leads to

$$\mathbf{j} = \frac{(1+\lambda B^2)(\mathbf{E}+\mathbf{q}\times\mathbf{B})-\kappa(\mathbf{E}+\mathbf{q}\times\mathbf{B})\times\mathbf{B}+[\kappa^2+\lambda(1+\lambda B^2)][(\mathbf{E}+\mathbf{q}\times\mathbf{B}).\mathbf{B}]\mathbf{B}}{(1+\lambda B^2)^2+\kappa^2B^2} \tag{2.10}$$

as the generalized Ohm's law with Hall current and ion slip.

In the present two-dimensional case, where the applied magnetic field is normal to the plane of the flow we may also take the induced field to be normal, so that (2.6 b) is automatically satisfied. Then since $\mathbf{q}$ is in the plane, the last term in $\mathbf{j}$ vanishes. Furthermore, in view of (2.5 b) and (2.6 a) we may introduce a stream function and an electrostatic potential by the relations

$$\mathbf{q} \equiv \nabla\times\mathbf{k}\psi \equiv \nabla\psi\times\mathbf{k}, \qquad \mathbf{E} \equiv -\nabla\phi, \tag{2.11 a, b}$$

where $\mathbf{k}$ is the unit vector normal to the flow plane. If we also introduce $B$, the magnitude of the magnetic field, by

$$\mathbf{B} = \mathbf{k}B, \tag{2.12}$$

then the equation for $\mathbf{j}$ becomes

$$\mathbf{j} = -\frac{(1+\lambda B^2)(\nabla\phi+B\nabla\psi)-\kappa B(\nabla\phi+B\nabla\psi)\times\mathbf{k}}{(1+\lambda B^2)^2+\kappa^2B^2}. \tag{2.13}$$

In view of (2.12) we also find from (2.6 c) that

$$\nabla\times B\mathbf{k} = \nabla B\times\mathbf{k} = R_m\mathbf{j}, \tag{2.14}$$

and of course, in general, from (2.6 c)*

$$\nabla.\mathbf{j} = 0. \tag{2.15}$$

The nature of the body force in the present two-dimensional case is of interest. Let us break up the magnetic field into two parts. Let $\mathbf{B}_A$ be the applied field produced by applied currents, as for example the constant field inside the ellipse in the present problem. Let $\mathbf{B}_I$ be the induced field, which may be looked on as produced by the induced currents according to (2.14). The body force on the fluid involves only the induced currents, since they are the ones flowing in the gas. Therefore, the body

* This equation could, of course, also be derived directly from charge conservation. Sources and sinks for gas current could exist in the flow if electrodes were introduced. If the current in both the gas and the electrodes is considered it must be divergence free. However, the gas current itself would appear to have sources or sinks at the electrode surfaces. In this paper we will not consider electrodes, although the work here could easily be extended to cover this case.

force may be written with the help of (2.12) and (2.14) as

$$(\nabla\times\mathbf{B}_I)\times(\mathbf{B}_I+\mathbf{B}_A)/R_m = -\nabla B_I^2/2R_m - B_A\,\nabla B_I/R_m.$$

This shows that in the two-dimensional case the body force due to the induced field is the gradient of a scalar, and wherever the applied field is constant, the whole body force is such a gradient.

The Euler equation (2.5 a) becomes

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = \tfrac{1}{2}\nabla q^2 - \mathbf{q}\times\mathbf{\Omega} = -\nabla(p+SB_I^2/2R_m) - SB_A\,\nabla B_I/R_m. \quad (2.16\,\text{a})$$

Thus, in the regions where $B_A$ is a constant, we have the usual hydrodynamic case of body forces derivable from a potential $SB_I(B_A+\frac{1}{2}B_I)/R_m$, with a Bernoulli constant along each streamline. Further by taking the curl of (2.16 a), with due regard for the two-dimensional nature of the flow, we find that

$$(\mathbf{q}\cdot\nabla)\,\mathbf{\Omega} = 0, \qquad \text{where } B_A = \text{constant}. \quad (2.16\,\text{b})$$

Here we have recognized that $\mathbf{\Omega}$ has only one component and put

$$\mathbf{\Omega} \equiv \nabla\times\mathbf{q} \equiv \Omega\mathbf{k}, \qquad \Omega = -\nabla^2\psi. \quad (2.17\,\text{a, b})$$

The last relation, coming from (2.11 a), is the usual one between the vorticity and stream function in plane incompressible flow. Equation (2.16 b) expresses the familiar fact that the vorticity is constant along streamlines. In the present case, it holds everywhere except at the ellipse $C_e$, across which $B_A$ is discontinuous, so the last term on the right of (2.16 a) cannot be expressed as a gradient. Therefore, vorticity is generated only on $C_e$, where $B_A$ jumps, and nowhere else in the flow.

To determine how much vorticity is generated on $C_e$, consider for the moment an applied field $B_A$ which varies rapidly but continuously near $C_e$. Rewrite the last term on the right of (2.16 a) in its original form $S\mathbf{j}\times\mathbf{B}_A$; where $\mathbf{j}$ is understood to be the induced current. Then take the curl and so obtain

$$\nabla\times[(\mathbf{q}\times\mathbf{\Omega})+S(\mathbf{j}\times\mathbf{B}_A)] = 0.$$

Since the tangential component of a curl-free vector must be continuous across a surface, the jump in vorticity across the surface is defined by requiring that the tangential component of

$$\mathbf{q}\times\mathbf{\Omega}+S(\mathbf{j}\times\mathbf{B}_A) \quad (2.18)$$

be continuous.

The other boundary conditions on $C_e$ may be obtained by similar considerations. Since the electric field is also curl-free its tangential component must also be continuous. The induced current density in the gas does not become infinite anywhere. The induced magnetic field is therefore continuous everywhere. In other words the jump in total magnetic field at $C_e$ is just equal to the jump in the applied field. The finite current also implies the absence of infinite forces. The velocity and pressure are, therefore, also continuous. In the absence of electrodes the gas current is everywhere divergence-free and, therefore, the normal component of current is continuous on $C_e$.

In summary, then, the problem of determining the hydrodynamic and electromagnetic fields consists of solving

$$\nabla \cdot \mathbf{j} = 0, \qquad \nabla B \times \mathbf{k} = R_m \mathbf{j}, \qquad (\mathbf{q} \cdot \nabla)\Omega = 0,$$

both inside and outside $C_e$. On $C_e$, where the applied field has a discontinuity, but there is no body, we require:

$$\mathbf{q},\ j_\nu,\ \partial\phi/\partial s,\ B_I \text{ are continuous on } C_e, \tag{2.19a}$$

where $\partial/\partial s$ denotes differentiation along the contour and $\nu$ the outer normal to $C_e$. Far from $C_e$, the induced fields must vanish, except possibly the velocity in the 'wake' of the contour. Let $\mathbf{l}_1$ be a unit vector in the direction of the free stream flow. Then

$$(\mathbf{q}-\mathbf{l}_1,\ \mathbf{j},\ \mathbf{E},\ B_I) \to 0 \text{ at infinity, except in the wake.} \tag{2.19b}$$

The remaining condition defines the vorticity jump on $C_e$ by continuity of the tangential component of (2.18),

$$\Omega_o - \Omega_i = S(j_\nu/q_\nu)B_{Ai} \text{ on } C_e, \tag{2.20}$$

where the subscript $i$ means inside $C_e$. To complete the formulation, one must use the relations (2.11), (2.13), and (2.17) between $\Omega$, $\psi$, $\phi$ and $\mathbf{j}$.

The set of equations (2.14)–(2.16) are non-linear in $\psi$ and $B$. In order to obtain approximate analytical solutions a perturbation technique will be used, valid for small values of the dimensionless parameters $S$ and $R_m$.

The quantities of main interest are the electromagnetic force and moment on the solenoid, which are equal and opposite to those on the fluid. Let $\mathbf{F}^*$ and $\mathbf{M}^*$ be the force and moment on the solenoid, which arise solely from the body force $\mathbf{j} \times \mathbf{B}$ on the fluid. Then

$$\mathbf{F} \equiv \frac{\mathbf{F}^*}{\sigma q^0 \mathscr{L}^2 (B^0)^2} = \frac{\mathbf{F}^*}{\rho (q^0)^2 \mathscr{L} S} = -\iint (\mathbf{j} \times \mathbf{B})\, dA, \tag{2.21}$$

$$\mathbf{M} \equiv \frac{\mathbf{M}^*}{\sigma q^0 \mathscr{L}^3 (B^0)^2} = -\iint \mathbf{r} \times (\mathbf{j} \times \mathbf{B})\, dA = \iint (\mathbf{r} \cdot \mathbf{j})\, dA, \tag{2.22}$$

where the area integrals are taken over the whole plane and $\mathbf{r}$ is the dimensionless position vector of a point so that $\mathbf{M}$ is taken about the origin of coordinates. By analogy with wing-section theory, one might be interested in the drag amd lift of the solenoid, that is, the forces parallel and perpendicular to the stream direction, respectively. These may be defined as

$$D = \frac{D^*}{\sigma q^0 \mathscr{L}^2 (B^0)^2} = \mathbf{F} \cdot \mathbf{l}_1, \tag{2.23}$$

$$L = \frac{L^*}{\sigma q^0 \mathscr{L}^2 (B^0)^2} = \mathbf{F} \cdot (\mathbf{k} \times \mathbf{l}_1) = \mathbf{F} \cdot \mathbf{n}_1, \tag{2.24}$$

where $\mathbf{l}_1$ and $\mathbf{n}_1$ are unit vectors along and normal to the stream direction, as shown in figure 1.

## 3. Perturbation expansion

All dependent variables are expanded in a double power series in $S$ and $R_m$. Zero-order terms represent the impressed fluid motion and magnetic field. In the present system of units, the zero-order velocity is a unit vector $\mathbf{l}_1$ in the direction of the free stream. The impressed field is normal to the flow plane and has magnitude unity inside the elliptical contour $C_e$ and zero outside it. The symbol $I$ will be used for this discontinuous quantity. Then we may write

$$\begin{aligned}
\mathbf{q} &= \mathbf{l}_1 + \mathbf{q}'S + \mathbf{q}^a R_m + \dots,\\
\psi = \mathbf{k}\times\mathbf{q} &= \psi^0 + \psi' S + \psi^a R_m + \dots,\\
\mathbf{\Omega} = \nabla\times\mathbf{q} &= \Omega' S + \Omega^a R_m + \dots,\\
B &= I + B'S + B^a R_m + \dots,\\
\mathbf{j} &= \mathbf{j}^0 + \mathbf{j}'S + \mathbf{j}^a R_m + \dots,\\
\phi &= \phi^0 + \phi' S + \phi^a R_m + \dots,\\
\mathbf{F} &= \mathbf{F}^0 + \mathbf{F}'S + \mathbf{F}^a R_m + \dots,\\
\mathbf{M} &= \mathbf{M}^0 + \mathbf{M}'S + \mathbf{M}^a R_m + \dots .
\end{aligned}$$

If these expansions are inserted into (2.13)–(2.16) and (2.18)–(2.22) and terms containing equal powers of $S$ and $R_m$ equated separately, we obtain the following results.

*Zero order*

$$\mathbf{j}^0 = -\frac{(1+\lambda I^2)(\nabla\phi^0 + I\nabla\psi^0) - \kappa I(\nabla\phi^0 + I\nabla\psi^0)\times\mathbf{k}}{d}, \tag{3.1a}$$

$$\nabla\cdot\mathbf{j}^0 = -(1+\lambda I^2)(\nabla^2\phi^0 + I\nabla^2\psi^0)/d = 0. \tag{3.1b}$$

Here $d$ is a constant which has different values inside and outside $C_e$. It is defined by

$$d \equiv (1+\lambda I^2)^2 + \kappa^2 I^2. \tag{3.2}$$

$$j^0_\nu,\ \partial\phi^0/\partial s \text{ are continuous on } C_e, \tag{3.1c}$$

$$\mathbf{F}^0 = -\iint (\mathbf{j}^0\times\mathbf{k}I)\,dA = \mathbf{k}\times\iint_i \mathbf{j}^0_i\,dA, \tag{3.1d}$$

$$\mathbf{M}^0 = \iint I\mathbf{k}(\mathbf{r}\cdot\mathbf{j}^0)\,dA = \mathbf{k}\iint_i (\mathbf{r}\cdot\mathbf{j}^0_i)\,dA. \tag{3.1e}$$

*First order in S*

Here we first use (2.14) to obtain

$$\nabla B'\times\mathbf{k} = 0.$$

Since $B'$ is independent of the variable normal to the plane, this indicates that $B'$ is at most a constant. Since $B'$ is continuous on $C_e$ and vanishes at infinity, it must vanish everywhere.

With $B' = 0$ we then have

$$\mathbf{j}' = -\frac{(1+\lambda I^2)(\nabla\phi' + I\nabla\psi') - \kappa I(\nabla\phi' + I\nabla\psi')\times\mathbf{k}}{d}, \tag{3.3a}$$

$$\nabla \cdot \mathbf{j}' = -(1+\lambda I^2)(\nabla^2\phi' + I\nabla^2\psi')/d = 0, \tag{3.3 b}$$

$$(\mathbf{l}_1 \cdot \nabla)\Omega' = \nabla \times (\mathbf{j}^0 \times \mathbf{k}I) = I(\mathbf{k}\cdot\nabla)\mathbf{j}^0 - I\mathbf{k}\nabla\cdot\mathbf{j}^0 = 0. \tag{3.3 c}$$

$$q',\ j'_\nu,\ \partial\phi'/\partial s \ \text{ are continuous on } C_e, \tag{3.3 d}$$

and
$$\Omega'_o - \Omega'_i = j^0_\nu / l_{1\nu}, \tag{3.3 e}$$

where the subscripts $o$, $i$ denote the regions outside and inside $C_e$, respectively.

$$\mathbf{F}' = -\iint (\mathbf{j}' \times \mathbf{k}I)\, dA = \mathbf{k} \times \iint_i \mathbf{j}'_i\, dA, \tag{3.3 f}$$

$$\mathbf{M}' = \iint I\mathbf{k}(\mathbf{r}\cdot\mathbf{j}')\, dA = \mathbf{k}\iint_i (\mathbf{r}\cdot\mathbf{j}'_i)\, dA. \tag{3.3 g}$$

*First order in $R_m$*

Here we first consider (2.16) and (2.20) which show that

$$(\mathbf{l}_1 \cdot \nabla)\Omega^a = 0,$$
$$\Omega^a_o - \Omega^a_i = 0 \quad \text{on } C_e.$$

Thus $\Omega^a$ is constant along zero-order stream lines with no jump at $C_e$. Since $\Omega^a$ is zero where the streamlines originate, it is zero everywhere, so

$$\boldsymbol{\Omega}^a = \nabla \times \mathbf{q}^a = 0.$$

Since we also have $\nabla\cdot\mathbf{q}^a = 0$ from (2.5 b), and since $\mathbf{q}$ is not discontinuous, $\mathbf{q}^a$ is constant everywhere. This constant must be zero to satisfy the condition (2.19 b) at infinity. Therefore $\nabla\psi^a = 0$.

Now from (2.13) one finds $\mathbf{j}^a$:

$$\mathbf{j}^a = -\frac{(1+\lambda I^2)\nabla\phi^a - \kappa I\nabla\phi^a \times \mathbf{k} + \mathbf{V}^0 B^a}{d}. \tag{3.4 a}$$

Here $\mathbf{V}^0$ is a vector which is made up entirely of zero-order quantities, and which will be defined later. It is different inside and outside the ellipse.

$$\nabla\cdot\mathbf{j}^a = -\frac{(1+\lambda I^2)\nabla^2\phi^a + \mathbf{V}^0\cdot\nabla B^a + B^a\nabla\cdot\mathbf{V}^0}{d} = 0, \tag{3.4 b}$$

$$\nabla B^a \times \mathbf{k} = \mathbf{j}^0. \tag{3.4 c}$$

$$j^a_\nu,\ \partial\phi^a/\partial s \ \text{ are continuous on } C_e. \tag{3.4 d}$$

$$\mathbf{F}^a = \mathbf{k} \times \iint_i \mathbf{j}^a_i\, dA - \iint (\mathbf{j}^0 \times \mathbf{B}^a)\, dA, \tag{3.4 e}$$

$$\mathbf{M}^a = \mathbf{k}\iint_i (\mathbf{r}\cdot\mathbf{j}^a_i)\, dA - \iint \mathbf{r} \times (\mathbf{j}^0 \times \mathbf{B}^a)\, dA. \tag{3.4 f}$$

Equations (3.1), (3.3) and (3.4) describe the present problem completely to the first order in the parameters $S$ and $R_m$, and their solution will be presented in the next section. Readers who wish to do so may pass on to the sections entitled Results and Discussion.

## 4. Solution of the perturbation equations

The solution of the perturbation equations is fairly straightforward but the algebraic details are quite lengthy. Here we shall omit most of the algebra. Readers who are interested in the details may refer to Kemp & Petschek (1958).

The basic equations to be solved are (3.1 b), (3.3 b) and (3.4 b) for the electrostatic potentials $\phi^0$, $\phi'$ and $\phi^a$. They are actually Poisson equations because in each case the terms other than the Laplacian of the potential are known. Either they are given, as in the case of $\psi^0$, or they are determined from zero-order quantities; $\psi'$ is found from $\mathbf{j}^0$ through $\Omega'$ by way of (3.3 e) and $B^a$ from $\mathbf{j}^0$ according to (3.4 c). Once the various $\phi$ are found the currents can be obtained from (3.1 a), (3.3 a) and (3.4 a) and then the forces and moments from (3.1 d, e), (3.3 f, g) and (3.4 e, f).

In actual practice it turns out to be easier to proceed slightly differently. The zero order and first order in $S$ cases satisfy identical differential equations and boundary conditions, except that $\psi^0$ is given while $\psi'$ must be determined, as pointed out above, from $\mathbf{j}^0$ through $\Omega'$. Therefore, the solution of (3.1 b) and (3.3 b) as Laplace equations in $\phi + I\psi$, can be carried out simultaneously. Then the appropriate values of $\psi$ can be inserted to complete the solution in each case, after $\psi'$ is found from the zero-order solution. On the other hand, to find $\phi^a$, equation (3.4 b) is treated as a Poisson equation with $\mathbf{V}^0$ and $B^a$ determined from zero-order quantities.

The natural coordinate system to use is an elliptic one so we introduce elliptic coordinates $\xi$, $\eta$ by the transformation

$$z = x + iy = re^{i\theta} = \epsilon a \cosh(\xi + i\eta) = \epsilon a \cosh \zeta. \tag{4.1}$$

Here $x$, $y$ are Cartesian coordinates parallel respectively to the major axis $2a$ and the minor axis $2b$ of the ellipse $C_e$ ($\xi = c$), and $\epsilon$ is the eccentricity given by

$$\epsilon^2 = 1 - h^2, \qquad h = b/a. \tag{4.2}$$

Each value of $\xi$ defines a member of a confocal family of ellipses with foci at $x = \pm a\epsilon$. On each ellipse $\eta$ is an angle running from 0 on the positive $x$-axis to $2\pi$.

Since we deal with Laplace and Poisson equations we must have available solutions of Laplace's equation in elliptic coordinates. The regions inside and outside the elliptical contour $C_e$, denoted respectively by subscripts $i$ and $o$, must be treated separately. Furthermore, since the gradients of $\phi$ and $\psi$ are physical quantities, they must be finite everywhere and vanish at infinity. Suitable solutions of Laplace's equation, which can be found by separation of variable in the elliptic coordinates, are

$$f_i = \alpha_o + \sum_1^\infty \alpha_m \cosh m\xi \cos m\eta + \beta_m \sinh m\xi \sin m\eta, \tag{4.3 a}$$

$$f_o = \gamma_o + \gamma_\xi \xi + \gamma_\eta \eta + \gamma_{\xi\eta} \xi\eta + \sum_1^\infty e^{-m\xi}(\gamma_m \cos m\eta + \delta_m \sin m\eta) \tag{4.3 b}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ are constants.

With these solutions available it is only a matter of algebra to find $\phi$ in terms of $\psi$ for the zero order and first order in $S$ cases, while for $\phi^a$ some simple particular solutions must also be constructed.

*Expressions for the zero order and first order in S potentials, forces and moments*

As pointed out above, the zero order and first order in $S$ equations for $\phi$ are the same and may be handled together. According to (3.1 b) and (3.3 b) they both satisfy

$$\nabla^2\phi + I\nabla^2\psi = 0, \tag{4.4}$$

and the boundary conditions that, on $C_e$, $j_\nu$ and $\partial\phi/\partial s$ are continuous. In both cases, $\mathbf{j}$ is also given by the same expression, (3.1 a) and (3.3 a).

A solution to (4.4) is obtained by using (4.3 a) for $\phi_i + \psi_i$ and (4.3 b) for $\phi_o$. The constants are found from the boundary conditions, and can be written in terms of the Fourier coefficients of $\partial\psi_i/\partial\eta$ on $C_e$. If we write

$$(\partial\psi_i/\partial\eta)_e = A_o + \sum_1^\infty A_m \cos m\eta + C_m \sin m\eta, \tag{4.5}$$

the constants are

$$\left.\begin{aligned}
\gamma_\xi = \gamma_{\xi\eta} = 0, \qquad \gamma_\eta &= A_o,\\
\alpha_m &= \{A_m \kappa - C_m[(1+\lambda)\coth mc + d_i]\}/m\Delta_m \cosh mc,\\
\beta_m &= \{A_m[(1+\lambda)\tanh mc + d_i] + C_m \kappa\}/m\Delta_m \sinh mc,\\
\gamma_m e^{-mc} &= \{A_m \kappa + C_m[1 + (1+\lambda)\tanh mc]\}/m\Delta_m,\\
\delta_m e^{-mc} &= \{-A_m[1 + (1+\lambda)\coth mc] + C_m \kappa\}/m\Delta_m,
\end{aligned}\right\} \tag{4.6}$$

where

$$\Delta_m \equiv 1 + d_i + (1+\lambda)(\tanh mc + \coth mc), \tag{4.7 a}$$

$$d_i \equiv (1+\lambda)^2 + \kappa^2. \tag{4.7 b}$$

The constants $\alpha_o$ and $\gamma_o$ are not significant and do not enter the equations.

The current vector, both inside and outside $C_e$, can be found from (3.1 a) or (3.3 a) by using (4.5) and (4.3), but is too lengthy to write here. When written in Cartesian vector components it has the form of a series, each term of which is the product of an exponential function of $\xi$ and a trigonometric function of $\eta$.

The forces and moments are obtained from (3.1 d, e) or (3.3 f, g) and require the integration of $\mathbf{j}_i\,dA$ and $\mathbf{r}.\mathbf{j}_i\,dA$ over the inside of the ellipse. Because of the form of $\mathbf{j}_i$, both these integrands also involve only terms which are the product of a function of $\xi$ and a function of $\eta$, so the integration is very simple. All but one term in each component vanishes because of the periodicity in $\eta$, and the results are

$$\mathbf{F} = -\pi a(\mathbf{x}_1 h\delta_1 - \mathbf{y}_1\gamma_1)e^{-c}, \tag{4.8 a}$$

$$\mathbf{M} = \tfrac{1}{2}\mathbf{k}\pi a^2\epsilon^2\gamma_2 e^{-2c}. \tag{4.8 b}$$

Now, in order to express $\mathbf{F}$ and $\mathbf{M}$ in terms of given parameters, it is only necessary to determine the Fourier coefficients $A_m$ and $C_m$ in the expansion of $(\partial\psi/\partial\eta)_e$, as indicated in (4.5).

*Zero-order current, force and moment*

For zero order the stream function is, of course, given as that of a uniform stream at angle $\alpha$ to the major axis of the ellipse:

$$\psi^0 = y\cos\alpha - x\sin\alpha = a\epsilon(\sinh\xi\sin\eta\cos\alpha - \cosh\xi\cos\eta\sin\alpha). \tag{4.9}$$

From (4.5) we see immediately that

$$A^0_m = C^0_m = 0, \quad m \neq 1; \qquad A^0_1 = b\cos\alpha, \qquad C^0_1 = a\sin\alpha. \tag{4.10}$$

When these are inserted into (4.6), the potential $\phi^0$ is completely determined (except for an additive constant which we shall ignore) by (4.3 a) for $\phi^0_i + \psi^0_i$ and by (4.3 b) for $\phi^0_o$.

$$\phi^0_i = \gamma^0_1 e^{-c} x/a + \delta^0_1 e^{-c} y/b, \tag{4.11 a}$$

$$\phi^0_o = e^{-\xi}(\gamma^0_1 \cos\eta + \delta^0_1 \sin\eta). \tag{4.11 b}$$

The current is likewise found from (3.1 a) to be

$$\mathbf{j}^0_i = \mathbf{x}_1 \gamma^0_1 e^{-c}/b + \mathbf{y}_1 \delta^0_1 e^{-c}/a \equiv j^0_{ix}\mathbf{x}_1 + j^0_{iy}\mathbf{y}_1, \tag{4.12 a}$$

$$\mathbf{j}^0_o = \frac{(\mathbf{x}_1\gamma^0_1 - \mathbf{y}_1\delta^0_1)(\cos\eta - e^{-2\xi}) + (\mathbf{x}_1\delta^0_1 + \mathbf{y}_1\gamma^0_1)\sin 2\eta}{2a\,\epsilon(\sinh^2\xi + \sin^2\eta)}. \tag{4.12 b}$$

This shows that the zero-order current inside the ellipse is a constant with Cartesian components

$$j^0_{ix} = \gamma^0_1 e^{-c}/b = [\kappa\cos\alpha + \sin\alpha(1+\lambda+h^{-1})]/\Delta_1, \tag{4.13 a}$$

$$j^0_{iy} = \delta^0_1 e^{-c}/a = [\kappa\sin\alpha - \cos\alpha(1+\lambda+h)]/\Delta_1, \tag{4.13 b}$$

where $\Delta_1$ is found from (4.7).

Since $\mathbf{j}^0_i$ is a constant, the calculation of the zero-order force and moment directly from (3.1 d, e) is simple:

$$\mathbf{F}^0 = (\mathbf{k}\times\mathbf{j}^0_i)\pi ab, \qquad \mathbf{M}^0 = 0. \tag{4.14}$$

These, of course, agree with (4.8). The lift and drag, normal and parallel to the stream velocity respectively, are easily found by resolving $\mathbf{j}^0_i$ along Cartesian coordinates $n$ and $l$ in those directions (figure 1). If we write

$$\mathbf{j}^0_i = \mathbf{l}_1 j^0_{il} + \mathbf{n}_1 j^0_{in} \tag{4.15}$$

we find from (4.13) and (4.14) that

$$L^0 = \mathbf{F}^0 . \mathbf{n}_1 = \pi ab j^0_{il} = \pi ab[\kappa + \sin\alpha\cos\alpha(h^{-1} - h)]/\Delta_1, \tag{4.16 a}$$

$$D^0 = \mathbf{F}^0 . \mathbf{l}_1 = -\pi ab j^0_{in} = \pi ab[1 + \lambda + h\cos^2\alpha + h^{-1}\sin^2\alpha]/\Delta_1. \tag{4.16 b}$$

Discussion of these results is postponed to a later section.

*First order in S velocity and stream function*

To find the first order in $S$ quantities one must know the stream function $\psi'$. This is determined by the vorticity $\Omega'$ according to (2.17 b). The vorticity is in turn determined from zero order quantities by (3.3 c, e). These equations are

$$\nabla^2\psi' = -\Omega', \qquad (\mathbf{l}_1 . \nabla)\Omega' = 0, \tag{4.17 a, b}$$

$$\Omega'_o - \Omega'_i = j^0_\nu / l_{1\nu} \text{ on } C_e. \tag{4.18}$$

Equation (4.17 b) indicates that $\Omega'$ is constant along zero-order streamlines, i.e. in the $l$-direction, and is thus a function of $n$ only. Equation (4.18) specifies the jump in $\Omega'$ at the ellipse caused by the discontinuity there in the applied field. The zero-order (free stream) streamlines which do not pass through the ellipse suffer no jump in vorticity, and since they come from a region with no vorticity, they never have any. In other words, the vorticity is confined to the inside of the ellipse and to the wake. The wake is bounded by the zero-order streamlines just tangent to the ellipse. In the $(l, n)$ coordinate system these streamlines are defined by

$$n = \pm N, \qquad N^2 = a^2(h^2\cos^2\alpha + \sin^2\alpha). \tag{4.19}$$

From (4.17) and (4.18), using (4.12a) for $\mathbf{j}_i^0$, we find the vorticity inside the ellipse $\Omega_i'$ and in the wake $\Omega_w'$ to be

$$\Omega_i' = W_c - W_v n/\surd(N^2-n^2), \qquad \Omega_w' = -2W_v n/\surd(N^2-n^2), \tag{4.20 a, b}$$

where the constants are

$$W_c \equiv -a^2(h^2 j_{ix}^0 \cos\alpha + j_{iy}^0 \sin\alpha)/N^2, \tag{4.20 c}$$

$$W_v \equiv a^2 h(j_{ix}^0 \sin\alpha - j_{iy}^0\cos\alpha)/N^2 = -a^2 h j_{in}^0/N^2. \tag{4.20 d}$$

With the vorticity known, the stream function follows by a straightforward superposition of the stream functions of elementary vortices of strength $\Omega'\, dndl$:

$$\psi' = -\frac{1}{2\pi}\iint_{i+w} \Omega'(\bar{n})\log|z-\bar{z}|\, d\bar{n}d\bar{l}. \tag{4.21}$$

Here $z = x+iy$ and $|z-\bar{z}|$ is the distance between the vortex and the field point.

One convenient way to find $\psi'$ from this formula is to first obtain $q_n' = -(\nabla\psi')_l$. The integral for $(\nabla\psi')_l$ can be converted by integration by parts and use of the divergence theorem into an integral on $C_e$ with $(\Omega_o^i - \Omega_i')$ in the integrand. Use of (4.20) then leads to

$$(\nabla\psi')_l = \frac{a}{2\pi}\int_0^{2\pi} (hj_{ix}^0\cos\bar{\eta} + j_{iy}^0\sin\bar{\eta})\log|z-\bar{z}_e|\, d\bar{\eta}.$$

The log term can be written in elliptical coordinates and expanded in a power series in $\exp[-(\zeta+\bar{\zeta}_e)]$ and $\exp[\pm(\zeta-\bar{\zeta}_e)]$, the plus sign being used inside the ellipse and the minus sign outside. The integration on $\bar{\eta}$ is then easily performed and the result, in the $(l, n)$ coordinates, is

$$q_{in}' = -\frac{l[j_{il}^0 - (1-h)j_{ix}^0\cos\alpha] + n[j_{in}^0 + (1-h)j_{ix}^0\sin\alpha]}{1+h}, \tag{4.22 a}$$

$$q_{on}' = -ahe^{-\xi}[j_{il}^0\cos(\eta-\alpha) + j_{in}^0\sin(\eta-\alpha)]/\epsilon. \tag{4.22 b}$$

To find $\psi'$ we can now integrate the relation $q_n' = -\partial\psi'/\partial l$. First we work outside the ellipse. $q_{on}'$ can be integrated with respect to $l$ if we notice from (4.1) that

$$\frac{de^{-\zeta}}{dz} = \tfrac{1}{4}a\epsilon(2\zeta + e^{-2\zeta}), \qquad l+in = ze^{-i\alpha}. \tag{4.23}$$

This integration determines $\psi_o'$ except for an undetermined function of $n$, which must be found from the vorticity relation (4.17a), and from the boundary conditions on $q_{lo}' = \partial\psi_o'/\partial n$. This function turns out to be zero outside the wake, but not inside it. The results for $q'_{lo}$ and $\psi'_o$ are

$$q'_{ol} = ahe^{-\xi}[j^0_{il}\sin(\eta-\alpha) - j^0_{in}\cos(\eta-\alpha)]/\epsilon + \\ + \begin{cases} 0 & \text{outside the wake,} \\ -2W_v\sqrt{(N^2-n^2)} & \text{inside the wake,} \end{cases} \tag{4.24}$$

$$\psi'_o = \tfrac{1}{4}ha^2[j^0_{il}(e^{-2\xi}\cos 2\eta + 2\xi) + j^0_{in}(e^{-2\xi}\sin 2\eta - 2\eta)] + \\ + \begin{cases} 0 & \text{outside the wake.} \\ -W_v N^2\left[\sin t\cos t + t - \begin{pmatrix}\frac{1}{2}\pi \text{ for } n > 0 \\ \frac{3}{2}\pi \text{ for } n < 0\end{pmatrix}\right] & \text{inside the wake.} \end{cases} \tag{4.25}$$

Here $t$ is an angle defined by

$$n/N \equiv \sin t, \qquad 0 \leqslant t \leqslant \tfrac{1}{2}\pi \text{ for } n > 0, \qquad \tfrac{3}{2}\pi \leqslant t \leqslant 2\pi \text{ for } n < 0. \tag{4.26}$$

An entirely similar procedure, beginning with $q'_{in}$, and keeping in mind the continuity of $\mathbf{q}'$ on $C_e$, leads to the velocity and stream function inside the ellipse:

$$q'_{il} = l\frac{[j^0_{in} + (1-h)j^0_{ir}\sin\alpha]}{1+h} - n\frac{[W_e + j^0_{il} - (1-h)j^0_{ix}\cos\alpha]}{1+h} - W_v\sqrt{(N^2-n^2)}, \tag{4.27}$$

$$\psi'_i = \tfrac{1}{2}(nq'_{il} - lq'_{in}) - \tfrac{1}{2}W_v N^2 t + K_i \tag{4.28}$$

where $K_i$ is a constant chosen so that $\psi'_o = \psi'_i$ on $C_e$. It differs for $n \lessgtr 0$.

As a check on the first order in $S$ flow field, the velocities far from the ellipse can be found. Because Bernoulli's equation holds everywhere but in the wake the pressure can also be found far from the ellipse except in the wake. An application of the integral momentum theorem to a large circle then yields the force and moment on the fluid due to the first order in $S$ flow field. It is found that this calculation exactly checks the zero-order force and moment. This shows the advantage of obtaining forces and moments from the body force, since to obtain them from the flow field requires a higher order calculation.

### *First order in S force and moment*

Knowing $\psi'$, the force and moment are found by returning to (4.8), where they are expressed in terms of $\gamma_1$, $\delta_1$ and $\gamma_2$. To determine these constants according to (4.6), the coefficients $A_m$, $C_m$ in the Fourier expansion of $\partial\psi'/\partial\eta$ on $C_e$ must be found for $m = 1, 2$.

Since $\psi'$ is continuous on $C_e$, either $\psi'_o$ or $\psi'_i$ may be used. When the expansion is written in the form of (4.5) we find that

$$A'_1 = -\frac{8}{3\pi}W_v Nha\cos\alpha = \left(-\frac{8}{3\pi}W_v N\right)A^0_1,$$

$$C'_1 = -\frac{8}{3\pi}W_v Na\sin\alpha = \left(-\frac{8}{3\pi}W_v N\right)C^0_1,$$

where (4.10) for the zero-order coefficients have been recalled. This result shows that $\mathbf{F}'$ is parallel to $\mathbf{F}^0$, and proportional to it:

$$\mathbf{F}' = \left(-\frac{8}{3\pi} W_v N\right)\mathbf{F}^0 = \frac{8}{3\pi}\frac{ahj^0_{in}\mathbf{F}^0}{\sqrt{(h^2\cos^2\alpha+\sin^2\alpha)}}. \qquad (4.29)$$

The second-order coefficients, which determine the moment, are

$$A_2' = -\frac{ha^2W_v}{1+h}(h\cos^2\alpha-\sin^2\alpha),\ C_2' = \frac{a^2}{2}\left[\frac{1+h^2}{2}W_v\sin 2\alpha - \frac{h(1-h)}{1+h}W_c\right].$$

When these are inserted into (4.6) for $\gamma_2 e^{-2c}$, and that into (4.8 b), we find that

$$\mathbf{M}' = -\mathbf{k}\frac{a^4\pi(1-h^2)}{8\Delta_2}\left\{\frac{2\kappa h}{1+h}W_v(h\cos^2\alpha-\sin^2\alpha)+\right.$$
$$\left.+[1+(1+\lambda)\tanh 2c]\left[\frac{1+h^2}{2}W_v\sin 2\alpha - \frac{h(1-h)}{1+h}W_c\right]\right\}. \qquad (4.30)$$

The quantities $W_v$ and $W_c$ are defined in (4.20c, d) and $\Delta_2$ in (4.7a). For a circle $h=1$, and we see that $\mathbf{M}'=0$, as required by symmetry.

*First order in $R_m$ field and potential*

As has been pointed out before, to the first order in $R_m$ there is a perturbed magnetic field, rather than a perturbed flow field as found to the first order in $S$. This field is related to $\mathbf{j}^0$ by (3.4c), from which, since $\mathbf{j}_i^0$ is a constant, it is easy to see that inside $C_e$

$$B_i^a = -j^0_{iy}x + j^0_{ix}y. \qquad (4.31\text{ a})$$

Outside $C_e$, $\mathbf{j}_o^0 = -\nabla\phi_o^0$ according to (3.1a) and (3.2). Using (4.11b) for $\phi_o^0$ and (4.13) for $\gamma_1^0$ and $\delta_1^0$, we find that

$$B_o^a = e^{-\xi}(\gamma_1^0\sin\eta - \delta_1^0\cos\eta) = ae^{c-\xi}(hj^0_{ix}\sin\eta - j^0_{iy}\cos\eta). \qquad (4.31\text{ b})$$

To find the potential $\phi^a$ we use (3.4b). The vector $\mathbf{V}^0$, which is obtained from the expansion of (2.13), is given by

$$\mathbf{V}^0 d \equiv (1+3\lambda I)\nabla\psi^0 + 2\lambda I\nabla\phi^0 - 2\kappa I\nabla\psi^0\times\mathbf{k} - \kappa\nabla\phi^0\times\mathbf{k} + 2\mathbf{j}^0 I[2\lambda(1+\lambda I^2)+\kappa^2]. \qquad (4.32)$$

This is constant inside $C_e$ since all zero-order terms have been found to be constant there. Outside $C_e$ it is variable, but $\nabla\cdot\mathbf{V}^0 = 0$, as can be seen from the definition (4.32). Therefore, $\phi^a$ satisfies the Poisson equation

$$\nabla^2\phi^a = -\mathbf{V}^0\cdot\nabla B^a/(1+\lambda I^2). \qquad (4.33)$$

This equation is solved both inside and outside, using (4.3) as the complementary solutions. The particular solutions are simple exponentials of $2\xi$ and trigonometric functions of $2\eta$, obtained using (4.31) and (4.32) and the zero-order quantities already found. The constants in the complementary solution are determined as before by matching $j_v^a$ and $\partial\phi^a/\partial\eta$ on $C_e$. This results in a set of simultaneous equations whose inhomogeneous parts (which come from the right side of (4.33)), have only constant terms, and terms second harmonic in $\eta$. The determinant of the coefficients is $d_i\Delta_m$, which does not in general vanish and we conclude that only $\alpha_o^a$, $\gamma_o^a$, $\gamma_\eta^a$ and the coefficients of the second harmonic terms are non-vanishing.

The resulting expressions for $\phi^a$ are

$$\phi_i^a = Ka^2\epsilon^2(\cosh 2\xi + \cos 2\eta) + \alpha_2^a \cosh 2\xi \cos 2\eta + \beta_2^a \sinh 2\xi \sin 2\eta, \qquad (4.34\,\mathrm{a})$$

$$\phi_o^a = \tfrac{1}{8}a^2\epsilon^2(K_\xi e^{-2\xi} - K_c \cos 2\eta + K_s \sin 2\eta) + e^{-2\xi}(\gamma_2^a \cos 2\eta + \delta_2^a \sin 2\eta) + \gamma_\eta^a. \qquad (4.34\,\mathrm{b})$$

The constants $K$, $K_c$ and $K_s$ come from the right-hand side of (4.33) and are defined below (5.9). The constant $K_\xi$, arising from the same source, is

$$K_\xi \equiv 2(a\epsilon)^{-2}(\gamma_1^{0^2} + \delta_1^{0^2})\kappa + (a\epsilon)^{-1}(\gamma_1^0 \cos\alpha - \delta_1^0 \sin\alpha).$$

$\gamma_2^a$, $\alpha_2^a$ and $\beta_2^a$ are related to $A_o^a$, $A_2^a$ and $C_2^a$ by (4.6); the latter two constants are defined below (5.9), and

$$A_o^a = \frac{(1+\lambda)abK_c - ab(j_{ix}^0 U_y - j_{iy}^0 U_x)}{2d_i} + \frac{a^2\epsilon^2(1-h)K_\xi}{2(1+h)},$$

with $U$ also defined below (5.9). $\gamma_2^a$ and $\delta_2^a$ are given by

$$\gamma_2^a e^{-2c} = \alpha_2^a \cosh 2c + \tfrac{1}{8}a^2\epsilon^2(K + K_c),$$

$$\delta_2^a e^{-2c} = \beta_2^a \sinh 2c - \tfrac{1}{8}a^2\epsilon^2 K_s.$$

*First order in $R_m$ current, force and moment*

The current is found by inserting (4.34) into (3.4 a) and using (4.31) for $B^a$. Inside $C_e$ the result is

$$\begin{aligned}-\mathbf{j}_i^a = K[(1+\lambda)(\mathbf{x}_1 x + \mathbf{y}_1 y) + \kappa(-\mathbf{x}_1 y + \mathbf{y}_1 x)]/2d_i + \overline{\mathbf{V}}_i^0(-j_{iy}^0 x + j_{ix}^0 y)/d_i + \\ + (4/a^2\epsilon^2 d_i)\{[(1+\lambda)\alpha_2^a - \kappa\beta_2^a](\mathbf{x}_1 x - \mathbf{y}_1 y) + \\ + [(1+\lambda)\beta_2^a + \kappa\alpha_2^a](\mathbf{x}_1 y + \mathbf{y}_1 x)\}. \qquad (4.35)\end{aligned}$$

The current outside could be found also, but we will now show that the force and moment do not depend on it.

The force and moment are given by (3.4 e, f). The second integral in each formula can be simplified by using the relation $\mathbf{j}^0 = \nabla \times \mathbf{B}^a$. The integrands become

$$\mathbf{j}^0 \times \mathbf{B}^a = -\tfrac{1}{2}\nabla(B^a)^2, \qquad \mathbf{r} \times (\mathbf{j}^0 \times \mathbf{B}^a) = \tfrac{1}{2}\nabla \times \mathbf{r}(B^a)^2.$$

Use of vector integral theorems for the divergence and curl then reduce the area integrals to line integrals which can easily be shown to vanish because $B^a$ is continuous across $C_e$ and vanishes as $\xi \to \infty$ (see (4.31 b)). The formulae for force and moment therefore reduce to

$$\mathbf{F}^a = \mathbf{k} \times \iint_i \mathbf{j}_i^a \, dA, \qquad \mathbf{M}^a = \mathbf{k} \times \iint_i (\mathbf{r} \cdot \mathbf{j}_i^a) \, dA.$$

They depend only on $\mathbf{j}_i^a$, not on the outside current $\mathbf{j}_o^a$.

The integrations are very simple. Since $\mathbf{j}_i^a$ is linear in $x$ and $y$, the symmetry of the ellipse indicates that

$$\mathbf{F}^a = 0. \qquad (4.36)$$

The only contributions to the moment come from $x^2$ and $y^2$ terms which arise when $\mathbf{r}\,.\,\mathbf{j}_i^a$ is formed from (4.33). After some reduction, the result is

$$\mathbf{M}^a = \mathbf{k}\,\frac{a^4\pi(1-h^2)}{4}\left\{\frac{h}{2}\,\frac{(\bar{V}^0_{ix}j^0_{iy}+\bar{V}^0_{iy}j^0_{ix})}{d_i}+\frac{A^a_2\kappa+C^a_2[1+(1+\lambda)2h/(1+h^2)]}{\Delta_2}\right\}. \tag{4.37}$$

$\bar{\mathbf{V}}^0$, which is closely related to $\mathbf{V}^0$, is defined below (5.9). Again the moment vanishes for a circle ($h = 1$), which is required by symmetry.

## 5. Results

The results for force and moment to the zeroth and first order in both $S$ and $R_m$ are collected here for convenience of reference and discussion.

*Zero order*

$$F^0 = \mathbf{l}_1 D^0 + \mathbf{n}_1 L^0. \tag{5.1}$$

From (4.2), (4.7) and (4.16) the lift and drag are

$$L^0 = \pi abj^0_{il} = \pi a^2 h[\kappa+\tfrac{1}{2}\sin 2\alpha(h^{-1}-h)]/\Delta_1, \tag{5.2a}$$

$$D^0 = -\pi abj^0_{in} = \pi a^2 h[1+\lambda+h\cos^2\alpha+h^{-1}\sin^2\alpha]/\Delta_1, \tag{5.2b}$$

where

$$\Delta_1 \equiv (1+\lambda)\,\frac{(1+h)^2}{h}+\lambda^2+\kappa^2. \tag{5.3}$$

Also from (4.14) we recall that

$$\mathbf{M}^0 = 0. \tag{5.4}$$

*First order in S*

From (4.29) and (5.2b) the force is related to $\mathbf{F}^0$ by

$$\mathbf{F}' = \frac{8}{3\pi}\,\frac{ah\mathbf{F}^0 j^0_{in}}{\sqrt{(h^2\cos^2\alpha+\sin^2\alpha)}} = -\frac{8}{3\pi}\,\frac{ah[1+\lambda+h\cos^2\alpha+h^{-1}\sin^2\alpha]}{\sqrt{(h^2\cos^2\alpha+\sin^2\alpha)}}\,\mathbf{F}_o$$
$$= -\frac{8}{3\pi^2 a}\,\frac{\mathbf{F}^0 D^0}{\sqrt{(h^2\cos^2\alpha+\sin^2\alpha)}}. \tag{5.5}$$

From (4.30), (4.20c, d) and (4.13) the moment can be put in the form

$$\frac{\mathbf{M}'}{\mathbf{k}} = -\frac{a^4\pi(1-h)(1+h)^3}{8\Delta_1\Delta_2(1+h^2)}\left\{\frac{\sin 2\alpha}{2}\left[(1+h^2)+\frac{2h^2}{h^2\cos^2\alpha+\sin^2\alpha}\right]+\right.$$
$$+\frac{h}{(1+h)^2}\left[(\lambda\sin 2\alpha+\kappa\cos 2\alpha)\,\times\right.$$
$$\left.\left.\times\left((1+h^2)+\frac{2h^2(1+\lambda)}{h^2\cos^2\alpha+\sin^2\alpha}\right)+\frac{\lambda h(1+h)^2\sin 2\alpha}{h^2\cos^2\alpha+\sin^2\alpha}\right]\right\}, \tag{5.6}$$

where

$$\Delta_2 = \frac{(1+\lambda)(1+h)^4}{2h(1+h^2)}+\lambda^2+\kappa^2. \tag{5.7}$$

*First order in $R_m$*

From (4.36) we may recall that

$$\mathbf{F}^a = 0. \tag{5.8}$$

The moment is too lengthy to write out in one expression. From (4.37) it is

$$\frac{\mathbf{M}^a}{\mathbf{k}} = \frac{\pi a^4(1-h^2)}{4}\left\{\frac{h}{2}\frac{\bar{V}^0_{ix}j^0_{iy}+\bar{V}^0_{iy}j^0_{ix}}{(1+\lambda)^2+\kappa^2}+\frac{A_2\kappa+C^a_2[1+(1+\lambda)2h/(1+h^2)]}{\Delta_2}\right\}, \tag{5.9}$$

where

$$A^a_2 = \frac{a^2(h^2j^0_{ix}U_x-j^0_{iy}U_y)-\frac{1}{2}\kappa a^2\epsilon^2K}{2(1+\lambda)^2+2\kappa^2}+\frac{a^2\epsilon^2K_s}{4},$$

$$C^a_2 = \frac{a^2h(j^0_{iy}U_x+j^0_{ix}U_y)}{2(1+\lambda)^2+2\kappa^2}+\frac{a^2\epsilon^2(K+K_c)}{4},$$

$$\mathbf{U} = [(1+\lambda)^2+\kappa^2]\bar{\mathbf{V}}^0_o-\bar{\mathbf{V}}^0_i,$$

$$\bar{\mathbf{V}} = [(1+\lambda I^2)^2+\kappa^2]^{-1}\{(1+3\lambda I)\nabla\psi^0+2\lambda I\nabla\phi^0-2\kappa I\nabla\psi^0\times\mathbf{k}-$$
$$-\kappa\nabla\phi^0\times\mathbf{k}+2\mathbf{j}^0I[2\lambda(1+\lambda I^2)+\kappa^2]\},$$

$$K_c = (a\epsilon)^{-1}(\delta^0_1\sin\alpha-\gamma^0_1\cos\alpha), \qquad K_s = (a\epsilon)^{-1}(\delta^0_1\cos\alpha+\gamma^0_1\sin\alpha),$$

$$K = -\bar{\mathbf{V}}^0_i.\mathbf{k}\times\mathbf{j}^0_i/(1+\lambda),$$

$$\frac{\gamma^0_1e^{-c}}{b} = \frac{\kappa\cos\alpha+\sin\alpha[1+\lambda+h^{-1}]}{\Delta_1}, \qquad \frac{\delta^0_1e^{-c}}{a} = \frac{\kappa\sin\alpha-\cos\alpha[1+\lambda+h]}{\Delta_1},$$

$$\nabla\phi^0_i = \mathbf{x}_1(\gamma^0_1e^{-c}/a)+\mathbf{y}_1(\delta^0_1e^{-c}/b), \qquad \nabla\psi^0 = -\mathbf{x}_1\sin\alpha+\mathbf{y}_1\cos\alpha,$$

$$\mathbf{j}^0_i = \mathbf{x}_1j^0_{ix}+\mathbf{y}_1j^0_{iy} = \mathbf{x}_1(\gamma^0_1e^{-c}/b)+\mathbf{y}_1(\delta^0_1e^{-c}/a).$$

For the special case of $\kappa=\lambda=0$, $\mathbf{M}^a$ reduces to the following very simple expression

$$\frac{\mathbf{M}^a}{\mathbf{k}} = \frac{\pi}{8}\frac{a^4h^2(1-h)}{1+h}\sin 2\alpha. \tag{5.10}$$

## 6. Discussion

The problem that has been considered basically contains six parameters; the interaction parameter $S$, the magnetic Reynolds number $R_m$, the Hall coefficient $\kappa$, the ion slip coefficient $\lambda$, the ratio of minor to major axis of the ellipse $h$, and the angle of attack $\alpha$. Since the analysis has been carried out as a perturbation in $S$ and $R_m$, the results for each order, as given in the previous section, are functions of the four parameters $\kappa$, $\lambda$, $h$ and $\alpha$. Discussion of these four-parameter functions is, necessarily, limited to consideration of certain limiting cases.

*Zero order*

To the zero order the gas current distribution and the resulting forces are computed from the undisturbed flow and magnetic field. Since to this order the hydrodynamic properties of the fluid do not enter, the zero-order solutions are not restricted to incompressible flows.

As may be seen from (4.12a) both the magnitude and the direction of the gas current inside the ellipse are independent of position. While this is true for all values of the parameters $\kappa$, $\lambda$, $h$ and $\alpha$, it cannot be generalized to all closed contours, but seems to be a property peculiar to ellipses. It can easily be demonstrated, for example, that it would not hold for a rectangular cross-section.

The fact that the zero-order moment vanishes for all ellipses follows directly from the fact that the current and, therefore, the body forces on the fluid are uniform throughout the ellipse.

Let us consider first the special case of high densities where $\kappa$ and $\lambda$ are both zero and discuss the force on ellipses one of whose axes is parallel to the flow ($\alpha = 0$ or $\frac{1}{2}\pi$). Equation (5.2a) shows that the lift is zero, and equation (5.2b) may be written in the dimensional form

$$D^{0*} = \sigma q^0 B^{0^2} \pi a^* b^* \left(\frac{H}{1+H}\right), \tag{6.1}$$

where $H$ is the ratio of the lengths of the axes perpendicular and parallel to the stream; i.e.,

$$H = h \text{ for } \alpha = 0, \qquad H = h^{-1} \text{ for } \alpha = \tfrac{1}{2}\pi.$$

The factor preceding the bracket is the drag the ellipse would have if there were no electric field to impede the current flow, i.e. if the current in the ellipse were $\sigma q^0 B^0$. The factor in brackets is then the reduction in drag due to the fact that an electric field is required to close the current loops outside the ellipse. In other words, it is a measure of the electrical impedance seen by the e.m.f. which is generated inside the ellipse as a result of the motion of the gas through the magnetic field. For ellipses of fixed area the drag increases monotonically as the length of the axis perpendicular to the stream increases.

The case of general angle of attack for $\kappa = \lambda = 0$ may be considered as the superposition of flows parallel and perpendicular to the major axis. Since the problem has been linearized, the total force is the vector sum of the forces resulting from the component flows. Because the drag per unit velocity is larger for the flow perpendicular to the major axis than it is for the flow parallel to it, the total force will not be a pure drag force but will also have a lift component.

In figures 2 and 3 the drag and lift for ellipses of different fineness ratios $h$ have been plotted as a function of angle of attack from (5.2). The perimeter $P$ has been used as the representative length $\mathscr{L}$ in this plot, so that the forces are compared for ellipses of the same perimeter. This choice was motivated by the fact that for a given magnetic field and a given mass of copper in the solenoid the joule dissipation required in the copper is a function only of the perimeter. On this basis, in addition to the factor due to the change in impedance with shape discussed above, the drag decreases as the ellipse becomes thinner because the area decreases.

The lift is zero for a circle because of symmetry, as mentioned above, and also zero for a very fine ellipse because the area goes to zero. It has a maximum for a fineness ratio of about 0·3. The ratio of lift to drag becomes infinite for infinitely thin ellipses at small angles of attack. At this point, of course, both the lift and the drag themselves approach zero. For a reasonable fineness ratio the lift-drag ratio is not large. For example, for a fineness ratio greater than 0·2 the lift is less than the drag at all angles of attack.

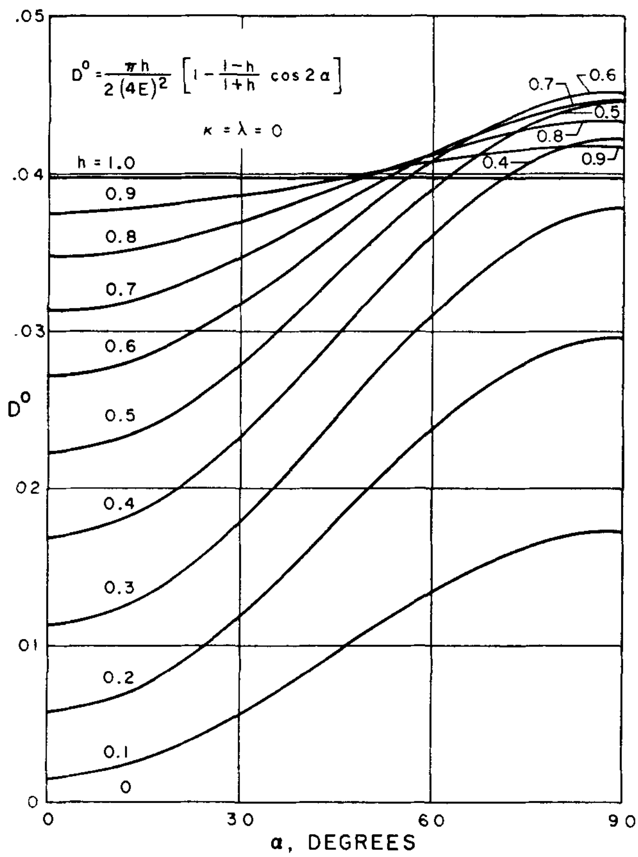


Figure 2. Non-dimensional zero-order drag $D^0 = D^{0*}/\sigma q^0 P^2 (B^0)^2$ (based on perimeter, $\mathscr{L} = P$) *vs* angle of attack $\alpha$ for varying fineness ratio $h$; plotted from equation (5.2 b) with Hall and ion slip coefficients $\kappa$ and $\lambda$ equal to zero. $E$ is the complete elliptic integral of the second kind.

In order to describe the effect of the Hall and ion slip coefficients $\kappa$ and $\lambda$ let us consider first a circular solenoid, $h = 1$. The Hall coefficient tends to produce a current in the stream direction. The cross product of this current with the magnetic field produces a lift even for a circle. Since the sign of the Hall current is independent of the sign of the magnetic field, the sign of the lift will depend on the sign of the magnetic field. Lift and drag have been plotted as a function of $\kappa$ in figure 4 from (5.2). In drawing this graph we have assumed that

$$\lambda = \kappa^2/500, \tag{6.2}$$

which is a limiting case of (A 5).

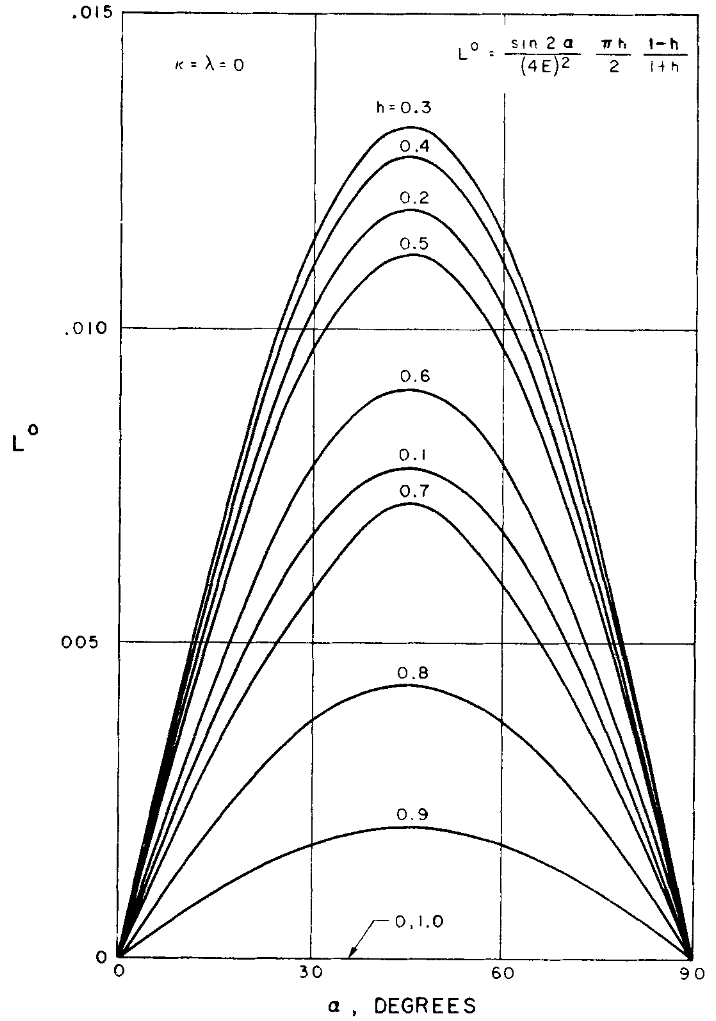


Figure 3. Non-dimensional zero-order lift $L^0 = L^{0*}/\sigma q^0 P^2 B^{0^2}$ (based on perimeter, $\mathscr{L} = P$) *vs* angle of attack $\alpha$ for varying fineness ratio $h$; plotted from equation (5.2 a) with Hall and ion slip coefficients $\kappa$ and $\lambda$ equal to zero. $E$ is the complete elliptic integral of the second kind.

As $\kappa$ increases from zero the lift initially increases and then decreases. However, both the drag and the total force decrease monotonically. Lift-drag ratios up to 7·9 can be obtained, but these large ratios occur when the total force has been reduced by almost a factor of ten compared with the case $\kappa = 0$.

As can be seen by looking at the general formula for zero-order drag (5.2 b) the reduction in drag associated with $\kappa$ depends on the geometry. It becomes less significant for flat ellipses. This reduction can probably also be made less significant by the use of electrodes whose potentials are adjusted to inhibit the flow of the Hall current. Non-uniform conductivity in the flow field, which would be encountered in an actual flight case, might also tend to produce a similar effect.

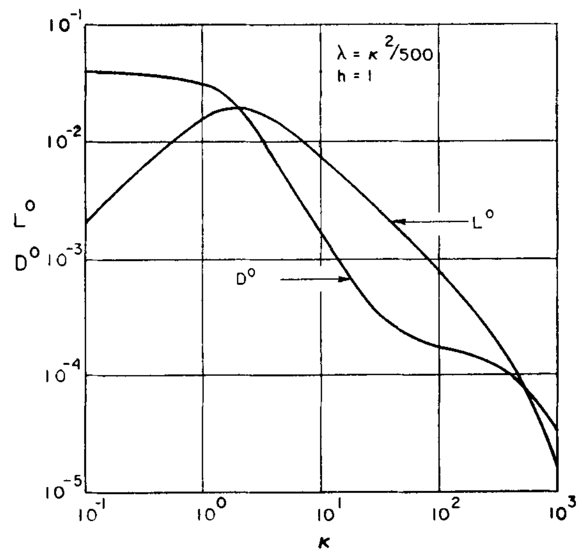


Figure 4. Non-dimensional zero-order lift $L^0$ and $D^0$ *vs* Hall coefficient $\kappa$ for a circle, $h = 1$; plotted from (5.2 a, b) with ion slip coefficient $\lambda = \kappa^2/500$.

It is instructive to examine two limiting values of $\kappa$ and $\lambda$ for general ellipses. Equation (6.2) shows that $\kappa$ can be large compared with unity while $\lambda$ is still small. This case will be approximated by $\kappa \to \infty$, $\lambda = 0$. In the second case examined, both $\kappa \to \infty$, $\lambda \to \infty$.

In the first case, the lift is much larger than the drag. The dimensional form of the lift may be obtained from (5.2 a) as

$$L^{0*} = \frac{\sigma q^0 B^{0^2} \pi a^* b^*}{\kappa} = (N_I e q^0) B^0 \pi a^* b^*. \tag{6.3}$$

In order to obtain (6.3), equations (A 2 b) and (2.8 a) have been used to define $\kappa$. Since the current inside the ellipse was shown to be constant, the force is the product of this current, the (constant) magnetic field, and the area. Thus, the term in parentheses in (6.3) must be the current. In order to produce this current the difference between the ion and electron velocities must be $q^0$. For $\lambda = 0$, (A 1 b) shows that the ion velocity is equal to the gas velocity $q^0$. The mean electron velocity must therefore be zero inside the ellipse. This is physically reasonable since, for $\kappa$ large, the electrons make tight spirals around the magnetic field lines. When the electron suffers a collision the centre of the spiral is moved by something less than the radius of the spiral. Since in this region the radius of the spiral is small and collisions are relatively infrequent, the electrons are practically stationary in the magnetic field. We may conclude that in the limit of $\kappa$ large but $\lambda$ still small the current may be approximated by assuming that in the magnetic field the electrons are stationary and the ions move with the gas velocity.

In the limit of both $\kappa$ and $\lambda$ large the drag becomes the dominant force. Using equations (A 2c) and (2.8 b), we find that the expression for zero-order drag becomes

$$D^{0*} = \frac{\sigma q^0 B^{0*} \pi a^* b^*}{\lambda} = (N_I N_N \epsilon_{IN} q^0) \pi a^* b^*. \tag{6.4}$$

From the definition of $\epsilon_{IN}$, the term in parentheses is the frictional force due to collisions between ions and neutrals when their relative velocity is $q^0$. In this case, then, the ions are held almost stationary in the magnetic field. It should be noted that in this limit the drag becomes independent of the magnetic field strength.

When the ion slip term is significant, the ion velocity is appreciably less inside than outside the ellipse. The assumption of uniform conductivity, which we have used throughout, implies that the ion concentration is uniform in space. In order to satisfy the continuity equation for the ions under these two conditions, a large portion of the ions entering the ellipse must recombine with electrons near the boundary. Also, where the streamlines leave the ellipse a corresponding number of ions must be produced. If the rates of ionization and recombination are sufficiently rapid, this requirement can be satisfied. The rapid rate of ionization would maintain thermal equilibrium everywhere. The uniform temperature which is implied by the assumption of incompressible flow then assures uniform degree of ionization. However, since ion slip occurs at low densities, the rate of recombination may not be sufficiently rapid to give thermal equilibrium everywhere. In this case the degree of ionization inside the ellipse would be greater than outside. Equation (6.4) would then suggest that the drag would be higher than one might expect assuming a uniform degree of ionization. The reduction in drag due to ion slip may, therefore, not be as large as the present calculations indicate, if the rates of ion recombination are not fast enough.

*First order in S*

To the first order in $S$ we have calculated first the modified flow field and then the resulting forces and moments. The perturbation velocity perpendicular to the free stream velocity is given in (4.22 a). The parallel component inside and outside the ellipse is given in (4.27) and (4.24). As was mentioned previously, for a circle the external flow outside the wake may be represented by a point source and vortex at the origin.

The gas pressure may be computed by substituting the first-order velocity and zero-order current and magnetic field into the momentum equation (2.5 a). It is interesting to note that in general the pressure gradient inside the ellipse is a constant, and its component in the free stream direction is negative. For a circle this pressure gradient is equal to one-half the zero-order magnetic force. For an ellipse, it is more complicated and is not in general parallel to the magnetic force.

The first-order forces (5.5) are of opposite sign to the zero-order forces. The fact that the forces must be reduced by the first-order terms is obvious since the flow velocity in the ellipse is reduced by the magnetic field. It is interesting to note, however, that the calculation shows that the first-order force is exactly anti-parallel to the zero-order force.

The magnitude of the first-order force may be used as a crude indication of the range of validity of the perturbation procedure. The ratio of first to zero-order forces is, from (5.5),

$$\frac{\mathbf{F}'S}{\mathbf{F}^0} = -\frac{8}{3\pi^2}(h^2\cos^2\alpha+\sin^2\alpha)^{-1/2}\frac{D^0S}{a}$$

$$= -\frac{8}{3\pi^2}(h^2\sin^2\alpha+\cos^2\alpha)^{-1/2}\left(\frac{D^{0*}}{a^*\rho q^{0^2}}\right). \quad (6.5)$$

The term in parentheses may be interpreted as a drag coefficient based on the major axis. If we take the square root as about unity, the first order in $S$ produces about a 25% correction when the zero-order drag coefficient is unity. One would therefore expect that the terms which have been calculated are insufficient beyond this point. It should be remembered that when $\kappa$ and $\lambda$ are not zero the drag coefficient may be small for fairly large values of $S$. The first-order terms become comparable with the zero-order terms only when the drag coefficient becomes comparable with unity, not when $S$ becomes comparable with unity.

The moment, which vanished to the zero order for ellipses, appears in the first-order approximation. In figure 5 the moment for $\kappa = \lambda = 0$, has been plotted from (5.6). The moment has been plotted in terms of the non-dimensional moment arm $M'^*/F^{0*}\mathscr{L}$ divided by the zero-order drag coefficient $2SD^0$. The plot therefore compares the moment arm for different ellipses with the same drag. It will be noted that the moment is zero at angles of attack of both 0° (major axis parallel to free stream) and 90° (minor axis parallel to free stream). At 0° the neutral point is unstable, so that the ellipse would orient itself at the other neutral point with the major axis perpendicular to the free stream. The sign of this moment may be explained crudely by noting that the flow through the downstream parts of the ellipse has already been decelerated somewhat in the magnetic field. The flow velocity is then higher in the upstream part of the ellipse and therefore the forces are larger there. A small angle between the major axis and the free stream direction therefore produces a moment which tends to increase the angle of attack.

For general $\kappa$ and $\lambda$ the moments become quite complex. One may to some extent regard the Hall effect as introducing another angle into the problem, the angle between the current and the electric field in a coordinate system moving with the gas. The introduction of this angle tends to change the angle of attack for a given moment. In order to illustrate this effect the stable angle of attack for an ellipse with $h = \frac{1}{3}$ has been plotted in figure 6 as a function of $\kappa$. This was calculated numerically from (5.6), and (6.2) was used to relate $\kappa$ and $\lambda$. It will be seen that for $\kappa$ both very

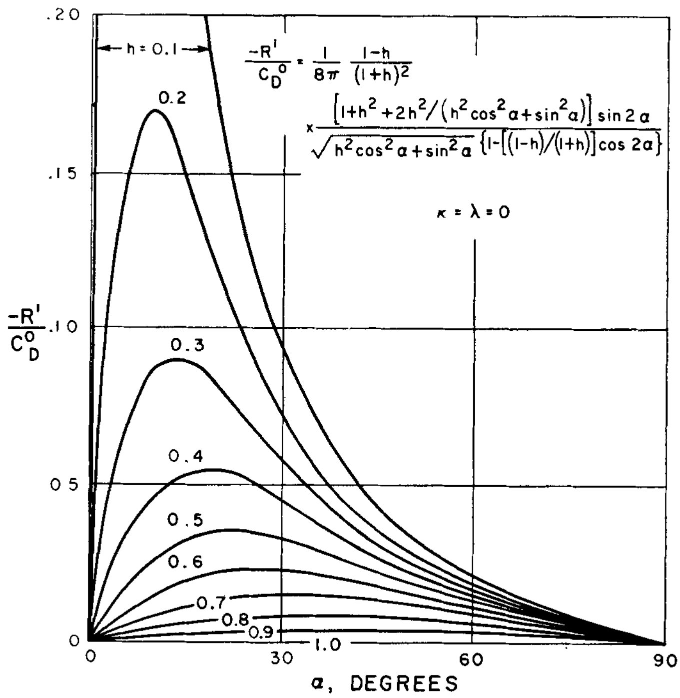


Figure 5. Non-dimensional first-order in $S$ moment arm $R' = M'/\mathscr{L}F^0$ divided by zero-order drag coefficient $C^0 = D^0/\frac{1}{2}\rho^0 q^{02}\mathscr{L} = 2SD^0$ *vs* angle of attack $\alpha$ for varying fineness ratio $h$. Plotted from equations (5.6) and (5.2 b) with Hall and ion slip coefficients $\kappa$ and $\lambda$ equal to zero.

small and very large the stable angle is 90°. For intermediate values the stable angle is reduced to almost 45°. In the limit $\kappa \to \infty$ and $\lambda = 0$ the stable angle would be 45°. For small values of $\kappa$ the stable angle depends on the ellipticity. However, for large $\kappa$ the curves for different ellipses approach one another asymptotically.

*First order in $R_m$*

The first-order calculations in $R_m$ principally determine the changes in the magnetic field caused by the flow. The change in field is given by (4.31 a) and (4.31 b). As a result of the uniform zero order current (4.12 a) inside the ellipse the field inside varies linearly with position, the field strength increasing toward the downstream side. For $\kappa$ and $\lambda$ zero the magnitude of the field change across the ellipse is roughly $R_m$ times the zero-order field. In this case the perturbed field is small compared with the applied field for $R_m$ less than unity. For $\kappa$ and $\lambda$ not zero the current and therefore the field change is reduced and the ratio of perturbed to applied fields remains small to larger values of $R_m$.

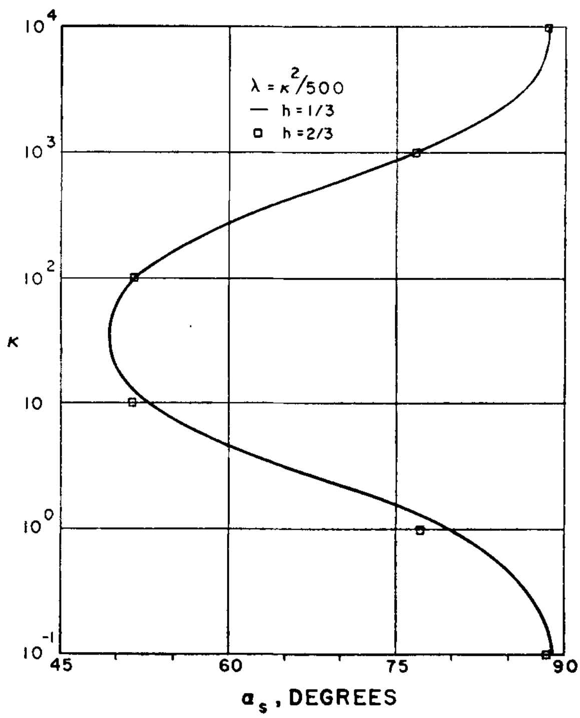


Figure 6. Angle of attack $\alpha_s$ at which ellipse has zero moment to the first order in $S$, and is stable with respect to that moment, *vs* Hall coefficient $\kappa$ for two fineness ratios $h = \frac{1}{3}, \frac{2}{3}$. Plotted from equation (5.6) with ion slip coefficient $\lambda = \kappa^2/500$.

The fact that the field variation is linear immediately implies that the first-order force is zero (5.8). The increase in force on the downstream side of the ellipse due to the increased magnetic field is exactly balanced to this order by the reduced force on the upstream side.

The moments to this order for $\kappa = \lambda = 0$ have been plotted in figure 7, from (5.9). The shape and magnitude are very similar to that for the first order terms in $S$. The sign of this moment is, however, opposite to that of $\mathbf{M}'$. Therefore, for $R_m$ somewhat larger than $S$ the stable angle of attack for the ellipse is 0°. This may be seen qualitatively from the argument above that the force on the downstream end is larger than on the upstream end.

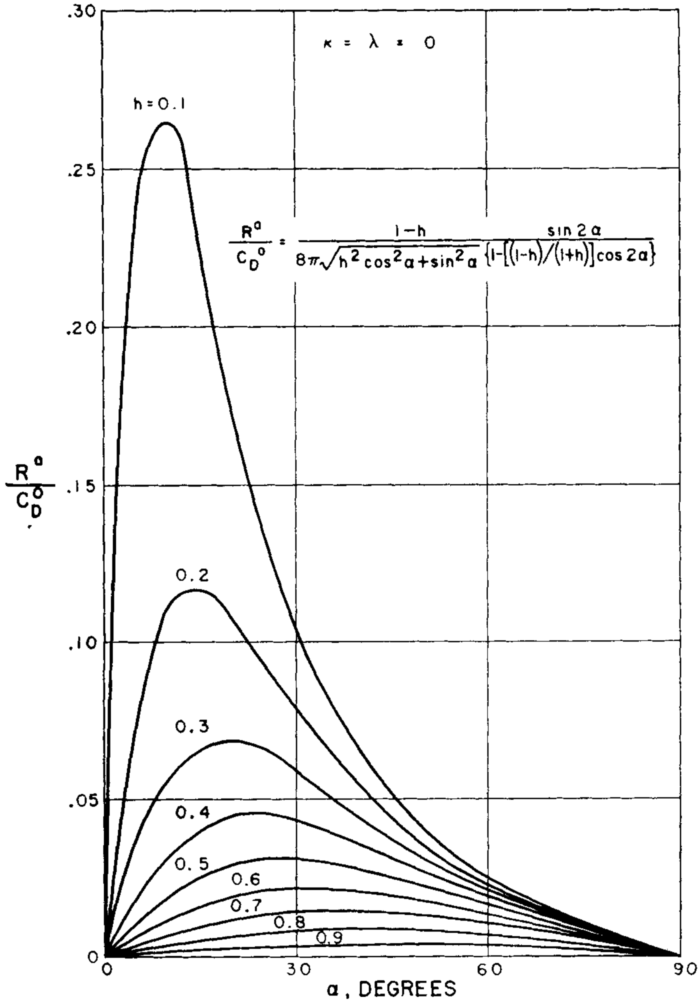


Figure 7. Non-dimensional first-order in $R_m$ moment arm $R^a = M^{a*}/\mathscr{L}F^0$ divided by zero-order drag coefficient $C^0 = D^0/\frac{1}{2}\rho q^{0^2}\mathscr{L} = 2SD^0$ *vs* angle of attack $\alpha$ for varying fineness ratio $h$. Plotted from equations (5.10) and (5.2 b) with Hall and ion slip coefficients $\kappa$ and $\lambda$ equal to zero.

## *Concluding remarks*

This analysis has been carried out as a step toward the solution of the more complicated situation which exists in hypersonic flight. Further work in this direction is presently under way at the AVCO Research Laboratory. Dr Frank Fishman is considering theoretically the corresponding perturbation problem for supersonic flow and Mr John Lothrop has begun an experimental investigation of the supersonic case utilizing a shock tube.

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## Appendix A. Generalized Ohm's law

The electrical conductivity of a partially ionized gas in the presence of a magnetic field has been calculated by Schluter (1950, 1951). A review of this work with some extensions and some algebraic corrections is given by Finkelnburg & Maecker (1956). Very briefly Schluter's approach consists of considering the separate momentum equations of the component gases, i.e. neutrals, ions, and electrons. These equations include, in addition to the acceleration, pressure gradient, and electromagnetic force terms, a frictional type of force arising from collisions between different components of the gas. This frictional force is proportional to the difference in mean velocity of the components. These three momentum equations may be combined to give the overall gas momentum equation (2.1 a) and in addition the mean velocities of the ions and electrons may be determined in terms of the local flow properties and their gradients. Instead of using the electron velocity explicitly it is more convenient to use the difference between electron and ion velocity, which is essentially the current.

The result (equations (51.17) and (51.18) in Finkelnburg & Maecker (1956) for the case of arbitrary degree of ionization of the gas is quite complex. However, for the special case of very small degrees of ionization where the partial pressures of the ions and electrons may be neglected the equations for current and ion velocity may be reduced to the form

$$\mathbf{j}^* = \sigma(\mathbf{E}^* + \mathbf{q}_I^* \times \mathbf{B}^* - \kappa^* \mathbf{j}^* \times \mathbf{B}^*), \tag{A 1 a}$$

$$\mathbf{q}_I^* = \mathbf{q}^* + \lambda^* \mathbf{j}^* \times \mathbf{B}^*, \tag{A 1 b}$$

where $\mathbf{q}_I^*$ is the mean ion velocity, and $\sigma$, $\kappa^*$ and $\lambda^*$ are coefficients which describe the gas conductivity. In terms of kinetic theory parameters

$$\sigma = \frac{N_I}{N_N} \frac{e^2}{\epsilon_{EN} + N_I \epsilon_{IE}/N_N}, \tag{A 2 a}$$

$$\kappa^* = 1/N_I e, \tag{A 2 b}$$

$$\lambda^* = 1/N_I N_N \epsilon_{IN}, \tag{A 2 c}$$

where the subscripts $N$, $I$, and $E$ refer to neutrals, ions, and electrons, respectively, $N$ is the number density of particles, $e$ is the absolute value of the electronic charge and $\epsilon_{jk}$ is a coefficient which describes the frictional force between species and is given by

$$\epsilon_{jk} = \frac{8}{3} \sqrt{\left(\frac{2}{\pi} kT \frac{m_j m_k}{m_j + m_k}\right)} Q_{jk} \tag{A 3}$$

where $k$ is Boltzmann's constant, $T$ is the temperature, $m$ is the particle mass and $Q_{jk}$ is the elastic collision cross-section.

The reduction of the general equations to the form given in (A 1) is not completely straightforward since it involves neglecting terms with gradients of the dynamic pressure of the ions. In the case being considered the discontinuous change in magnetic field produces a discontinuous change in ion velocity according to (A 1 b). In the immediate vicinity of this discontinuity the above equations are therefore not valid. The width of this region would, however, be of the order of one mean free path and, therefore, very small compared with the characteristic dimensions of the solenoid. Furthermore the total force due to the current resulting from the terms which have been dropped cannot be larger than the dynamic pressure of the ions. The equations (A 1) are, therefore, valid everywhere except in the immediate vicinity of the discontinuity in magnetic field and the discrepancies there do not produce significant effects on the flow field as a whole.

The generalized Ohm's law given in (2.3) may be obtained by direct substitution of (A 1 b) in (A 1 a). The physical significance of the coefficients is, however, more easily described in terms of the equations (A 1). By considering the case of zero magnetic field strength it is clear that $\sigma$ is the electrical conductivity in the absence of a magnetic field. From (A 1 b) it may be seen that the coefficient, $\lambda^*$, determines the magnitude of the slip velocity of the ions relative to the neutrals. The term involving $\kappa^*$ gives rise to a current which is not parallel to the electric field when observed from a coordinate system moving with the ion velocity. Using terminology carried over from solid state physics this current is usually referred to as a Hall current.

Using (A 2) and (A 3) it can easily be shown that the non-dimensional Hall coefficient $\kappa$, given by (2.8 a), is the product of the angular frequency at which the electrons spiral around the magnetic field lines and the mean free time between collisions of electrons with ions or neutrals. In other words the Hall coefficient is $2\pi$ times the average number of revolutions an electron experiences between collisions. The magnitude of the Hall coefficient depends, of course, on the physical conditions being considered. However, to obtain an estimate of the magnitude we will consider the case where the magnetic pressure $B^{0^2}/2\mu$ is equal to the gas pressure $p^*$. This condition is equivalent to taking both $S/R_m$ and $p^*/\rho q^{0^2}$ of order unity. Using this condition and further assuming that the degree of ionization is so low that the second term in the denominator of equation (A 2 a) is negligible and taking $Q_{EN} = 1 \times 10^{-15}\,\text{cm}^2$ one finds

$$\kappa \doteqdot 0{\cdot}2\sqrt{(\rho_{SL}/\rho)} \tag{A 4}$$

where $\rho_{SL}$ is the sea-level density of air. From the above formula it is clear that the Hall currents must be considered in the density regions that are of interest for high speed flight applications of magneto-hydrodynamics.

The ion slip coefficient $\lambda$ may be expressed in terms of the Hall coefficient as follows:

$$\lambda = \frac{\epsilon_{EN}}{\epsilon_{IN}}\left(1 + \frac{N_I}{N_N}\frac{\epsilon_{IE}}{\epsilon_{EN}}\right)\kappa^2 \doteqdot \frac{\kappa^2}{500}\left(1 + \frac{N_I}{N_N}\frac{\epsilon_{IE}}{\epsilon_{EN}}\right) \tag{A 5}$$

where the masses of the ions and neutrals are taken as the molecular weight of $N_2$ and it is assumed that $Q_{IN}/Q_{EN} = 3$. As pointed out by Patrick (1956), under typical flight conditions $\epsilon_{IE}/\epsilon_{EN}$ is of the order of 500. For degrees of ionization less than $2 \times 10^{-3}$ the term in brackets may therefore be taken as unity.

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